# Preparation of atomic Fock states by trap reduction

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We describe the preparation of atom-number states with strongly interacting bosons in one dimension or spin-polarized fermions. The procedure is based on a combination of weakening and squeezing of the trapping potential. For the resulting state, the full atom-number distribution is obtained. Starting with an unknown number of particles $N_i$, we optimize the sudden change in the trapping potential which leads to the Fock state of $N_f$ particles in the final trap. Nonzero temperature effects as well as different smooth trapping potentials are analyzed. A simple criterion is provided to ensure the robust preparation of the Fock state for physically realistic traps.

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## I. INTRODUCTION

Given the importance of photon statistics in quantum optics, the field of atom statistics is expected to develop vigorously in atom optics, fueled by the current ability to measure the number of trapped ultracold atoms with nearly single-atom resolution and without ensemble averaging of fluctuations [1–3]. Among the possible atomic distributions, pure atom-number (Fock) states form a fundamental basis and hold unique and simple properties that make them ideal for studies of quantum dynamics of few-body interacting systems [4], precision measurements [5], or quantum information processing [6–8]. Efficient and robust creation, detection, and manipulation of atom-number states are thus important goals in atomic physics. Several approaches have been proposed and explored recently with theoretical and experimental work leading to sub-Poissonian and, in the limit, number states, such as atomic tweezers [9,10], interferometric methods [4,11], Mott insulator states [12], or atomic culling [1,13]. None of these methods is so far fully satisfying if the individual atoms have to be addressed (a problem of the Mott insulator states in optical lattices), and if an arbitrary number of atoms is to be produced reliably and with small enough variance for the trapped atom number, so further research is still required.

In a previous paper [14] a method was proposed in which the trapping potential is simultaneously weakened and squeezed so that the final trap holds a desired number state. For a Tonks-Girardeau (TG) gas, it was shown that this mixed trap reduction yields optimal results even when the process is sudden. From the expression of the number variance, a simple criterion for optimal performance was obtained, namely, that the subspace spanned by the occupied levels in the initial trap configuration contains the subspace of the bound levels in the final trap. Starting from an unknown number of particles trapped at zero temperature, the mixed trap-reduction method assures that the final state indeed corresponds to the desired Fock state by avoiding the momentum or position space truncations inherent in pure squeezing or pure weakening (the latter being called "culling" in [13]).

In [13,14] the potential traps considered for simplicity were finite square wells so doubt could be cast on the validity of the results in actual smooth traps. Other limitations were the consideration of zero-temperature initial states, and a statistical analysis limited to the first and second moments of the number distribution. In this paper we overcome these shortcomings by studying the mixed trap-reduction process using smooth potentials, states with finite temperature, and the full number distribution.

## II. TONKS REGIME AND POLARIZED FERMIONS

The strongly interacting regime of ultracold bosonic atoms can be described by the so-called TG gas [15], which is achieved at low densities and/or large one-dimensional scattering length [16,17]. It has been argued [14] that this regime is optimal for the creation of atomic Fock states by mixed trap reduction.

The TG gas and its "dual" system of spin-polarized ideal fermions behave similarly, and share the same one-particle spatial density as well as any other local-correlation function, while differ on the nonlocal correlations.

The fermionic many-body ground-state wave function of the dual system is built at time $t=0$ as a Slater determinant for $N_i$ particles, $\psi_F(x_1,\ldots,x_{N_i})=\frac{1}{\sqrt{N_i!}}\det_{n,k=1}^{N_i}\varphi_n^i(x_k)$, where $\varphi_n^i(x)$ is the $n$th eigenstate of the initial trap, whose time evolution will be denoted by $\varphi_n(x,t)$ when the external trap is modified. The bosonic wave function, symmetric under permutation of particles, is obtained from $\psi_F$ by the Fermi-Bose (FB) mapping [15,18] $\psi(x_1,\ldots,x_{N_i})$

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$=\mathcal{A}(x_1,\ldots,x_{N_i})\psi_F(x_1,\ldots,x_{N_i})$, where $\mathcal{A}=\Pi_{1\le j<k\le N_i}\mathrm{sgn}(x_k-x_j)$ is the "antisymmetric unit function." Noting that $|\psi|^2=|\psi_F|^2$ it is clear that both systems obey the same counting statistics. Moreover, since $\mathcal{A}$ does not include time explicitly, the mapping is also valid when the trap Hamiltonian is modified, and the time-dependent density profile resulting from this change can be calculated as [19] $\rho(x,t)=N_i\int|\psi(x,x_2,\ldots,x_{N_i};t)|^2dx_2\cdots dx_{N_i}=\Sigma_{n=1}^{N_i}|\varphi_n(x,t)|^2$. By reducing the trap capacity (maximum number of bound states and thus particles that it can hold in the TG regime) some of the $N_i$ atoms initially confined may escape and only $N$ will remain trapped.

To determine whether or not sub-Poissonian statistics or a Fock state is achieved in the reduced trap we need to calculate the atom-number fluctuations.

## III. SUDDEN APPROXIMATION: FULL COUNTING STATISTICS

We shall now describe the preparation of Fock states by an abrupt change in the trap potential to reduce its capacity. Consider a trap with an unknown number of particles $N_i$, which supports a maximum of $C_i$ bound states. Generally $N_i$ is smaller than the capacity of the trap $C_i$. The trapping potential is abruptly modified to a final configuration of smaller capacity $C_f$. Similarly the final number of trapped particles will be $N_f\le C_f$. We are interested in the optimal potential change such that $N_f=C_f$ to prepare the atomic Fock state $|N_f=C_f\rangle$.

Let $\alpha=i,f$ stand for initial and final configurations. The Hilbert space associated with the Hamiltonian of a particle moving in any realistic trap $V_\alpha$ is the direct sum $\mathcal{H}_\alpha=\mathcal{B}_\alpha\oplus\mathcal{R}_\alpha$ of the subspace spanned by the bound states $\mathcal{B}_\alpha=\{|\varphi_j^\alpha\rangle|j=1,\ldots,C_\alpha\}$, and that of scattering states $\mathcal{R}_\alpha=\{|\chi_k^\alpha\rangle|k\in\mathbb{R}\}$. Consider the projector onto the final bound states $\mathcal{B}_f$ defined as

$$\hat{\Lambda}_f=\sum_{j=1}^{C_f}|\varphi_j^f\rangle\langle\varphi_j^f|. \tag{1}$$

Within the TG regime and for spin-polarized fermions, the asymptotic mean number and variance of trapped atoms are [14]

$$\langle N_f\rangle=\mathrm{Tr}(\hat{\Lambda}_i\hat{\Lambda}_f) \tag{2}$$

and

$$\sigma_{N_f}^2=\mathrm{Tr}[\hat{\Lambda}_i\hat{\Lambda}_f-(\hat{\Lambda}_i\hat{\Lambda}_f)^2], \tag{3}$$

where

$$\hat{\Lambda}_i=\sum_{n=1}^{N_i}|\varphi_n^i\rangle\langle\varphi_n^i| \tag{4}$$

is the projector onto the bound subspace occupied by the initial state. We may thus conclude that trap reduction can actually lead to the creation of Fock states with $\langle N_f\rangle=C_f$ and $\sigma_{N_f}^2=0$ quite simply when the initial states span the final ones,

$$\hat{\Lambda}_f\subseteq\hat{\Lambda}_i. \tag{5}$$

In fact the full atom-number distribution [20] is accessible in the atom culling experiments [1] and we next focus our attention on it. Consider the characteristic function of the number of particles in the bound subspace of the final trap,

$$F(\theta)=\mathrm{Tr}[\hat{\rho}e^{i\theta\hat{\Lambda}_f\hat{n}\hat{\Lambda}_f}]. \tag{6}$$

Following [21], the atom-number distribution can be obtained as its Fourier transform,

$$p(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-in\theta}F(\theta)d\theta, \tag{7}$$

with $n=1,\ldots,C_f$.

The characteristic function of spin-polarized fermions or a Tonks-Girardeau gas restricted to a given subspace was studied in [22–24]. Using the projector for the bound subspace in the final configuration,

$$F(\theta)=\det[1+(e^{i\theta}-1)\hat{\Lambda}_f\hat{\Lambda}_i]. \tag{8}$$

For computational purposes it is convenient to use the basis of single-particle eigenstates $|\varphi_m^f\rangle$ which spans the final bound subspace so that $F(\theta)=\det\mathbf{A}$, where $\mathbf{A}$ is a $C_f\times C_f$ matrix with elements

$$A_{nm}=\delta_{nm}+[\exp(i\theta)-1]\langle\varphi_n^f|\hat{\Lambda}_i|\varphi_m^f\rangle. \tag{9}$$

Clearly, if $\hat{\Lambda}_f\subseteq\hat{\Lambda}_i$, $A_{nm}=\exp(i\theta)\delta_{nm}$, $F(\theta)=\exp(i\theta C_f)$, and the Kronecker-delta atom-number distribution associated with the Fock state $|N=C_f\rangle$ is obtained,

$$p(n)=\delta_{n,C_f}. \tag{10}$$

For completeness we note that the cumulant-generating function $\ln F(\theta)$ admits the expansion

$$\ln F(\theta)=\sum_{n=1}^{\infty}\kappa_n\frac{(i\theta)^n}{n!}, \tag{11}$$

from which the mean $\kappa_1=\langle N_f\rangle$ in Eq. (2) and variance $\kappa_2=\sigma_{N_f}^2$ in Eq. (3) are just the first two orders.

Different regimes of interactions for ultracold Bose gases in tight waveguides can be characterized by a single parameter $\gamma=mg_{1D}L/\hbar^2N$, where $g_{1D}$ is the one-dimensional coupling strength, $L$ the size of the system, and $m$ and $N$ the mass and number of atoms, respectively. $\gamma$ can be varied [17] allowing to explore the physics from the mean-field regime ($\gamma\ll1$) to the TG regime ($\gamma\gg1$) [16]. We note that for the system to remain in the TG regime, it suffices to keep or decrease the density since

$$\gamma_f=\gamma_i\frac{n_i}{n_f}. \tag{12}$$

In particular, trap weakening clearly leads to a reduction in the density so that the system goes deeper into the TG regime.

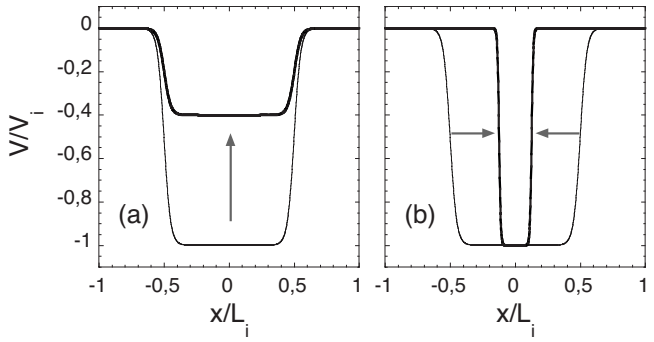


FIG. 1. Schematic potential change for (a) trap weakening and (b) squeezing, relative to the initial width $L_i$ and depth $V_i$, for $\sigma=0.05L$; see Eq. (13).

## IV. DEPENDENCE ON THE TRAPPING POTENTIAL

In this section we shall discuss the efficiency of the trap-reduction procedure at zero temperature, focusing on the relevance of the shape of the confining potential. In particular, instead of the idealized square potentials used in [13,14] we shall study here the family of "bathtub" potentials

$$\mathcal{V}_\alpha(x;V_\alpha,L_\alpha,\sigma_\alpha)=-\frac{1}{2}V_\alpha\left[1-\tanh\left(\frac{|x|-L_\alpha}{\sigma_\alpha}\right)\right], \quad (13)$$

as well as the inverted Gaussian potential

$$\mathcal{V}(x)=-V_\alpha e^{-x^2/2\delta^2}. \quad (14)$$

For the bathtub $L_\alpha$ and $V_\alpha$ play, respectively, the role of the width and depth of the trap, while $\sigma_\alpha$ is an additional parameter describing the smoothness of the potential trap.

The spectrum and eigenfunctions can be found numerically by a standard technique, first differencing the Hamiltonian and then diagonalizing the tridiagonal matrix obtained by such difference scheme [25].

For the bathtub potential, a given $U_\alpha=2mV_\alpha L_\alpha/\hbar^2$ and $\tilde{\sigma}_\alpha=\sigma_\alpha/L_\alpha$ define a family of isospectral potentials. In dimensionless units, their eigenvalues are the same and so are their eigenfunctions. In the limit of a square potential $U_\alpha \approx C_\alpha^2\pi^2$. For the Gaussian potential a single parameter $U_\alpha=V_\alpha\delta_\alpha^2$ defines an isospectral family.

Pure trap *weakening* corresponds to $V_f<V_i$, while $L_f=L_i$, and pure trap *squeezing* to $L_f<L_i$, keeping $V_f=V_i$ (Fig. 1). We shall next describe the efficiency of atomic Fock state preparation by mixed trap reduction ($V_f<V_i$ and $L_f<L_i$), keeping constant the relative smoothness parameter $\tilde{\sigma}_i=\tilde{\sigma}_f$, and going from the $U_i$ to the $U_f$ families of traps. This procedure allows us to apply squeezing of the potential up to any desired value of $L_f$ keeping its bathtub shape.

Figure 2 illustrates the full counting statistics of the resulting state in different limits of a trap-reduction scheme. Both pure weakening and squeezing fail to produce an atom-number state since the condition in Eq. (5) is not fulfilled. It was shown in [14] that this limitation arises as the result of truncation of the final state both in coordinate (pure weakening) or momentum (pure squeezing) space, with respect to the desirable Fock state $|N=C_f\rangle$. However, this state, whose full distribution reduces to a Kronecker delta $\delta_{n,C_f}$ [see Eq. (10)], can be obtained by combining both strategies, as shown in the middle panel. The final trap is then perfectly filled by the pure atom-number state.

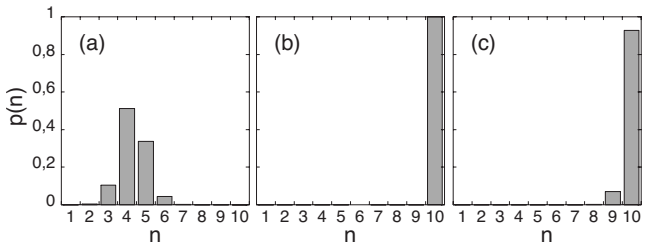


FIG. 2. Atom number distribution $p(n)$ in a trap-reduction setup combining both squeezing and weakening techniques, for a bathtub potential given by Eq. (13) with $\sigma_\alpha=0.03L_\alpha$ and the parameters of the trap being $N_i=C_i=100$ and $C_f=10$ ($U_i=10^4\pi^2$, $U_f=10^2\pi^2$). Plot (a) corresponds to almost pure squeezing, with $L_f/L_i=0.04$, while plot (c) corresponds to pure weakening, $L_f/L_i=1$, showing that it is the combination of both techniques, $L_f/L_i=0.5$, plot (b), the most efficient way for our purposes.

In what follows we shall characterize the efficiency of the method just by the mean and atom-number variance of the prepared state [see Eqs. (2) and (3)]. Let us consider different trap geometries. Generally, the effect of the smoothness of the potential is to increase the density of states near the brims, where the spacing between adjacent energy states is reduced (see Fig. 3). As a consequence, a higher control of the depth of the potential is required. Nonetheless, Fig. 4 shows that by increasing the smoothness, Fock state creation condition (5) is actually satisfied for a broader range of parameters which includes conditions nearer pure weakening and pure squeezing. This is because the initial state is spread out along the same region in configuration space as the final one; moreover the looser confinement reduces the momentum components of the final state, which can be resolved more easily by the initial state.

We might conclude that an invariably efficient strategy for the sudden transition between $U_i$ and $U_f$ trap families is achieved by reducing to half the width of the initial trap and reducing the depth to the trap accordingly, so as to achieve the desired $U_f$ and capacity $C_f$,

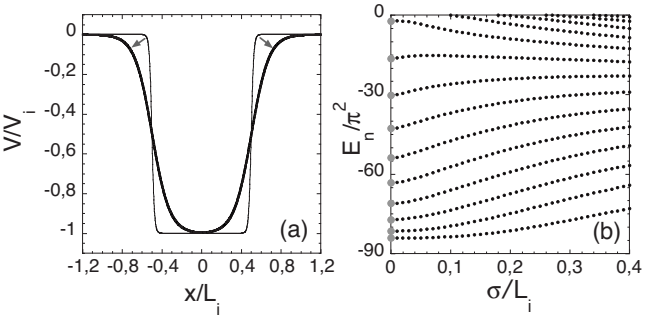


FIG. 3. Effect of the smoothness of the trapping bathtub potential (a) on the spectrum (b). As $\sigma$ increases the density of states concentrates near the brim of the trap and more bound states appear. The spectrum in the low panel is obtained for a potential $V=(10\pi)^2$ (in units of $\hbar^2/2mL^2$).

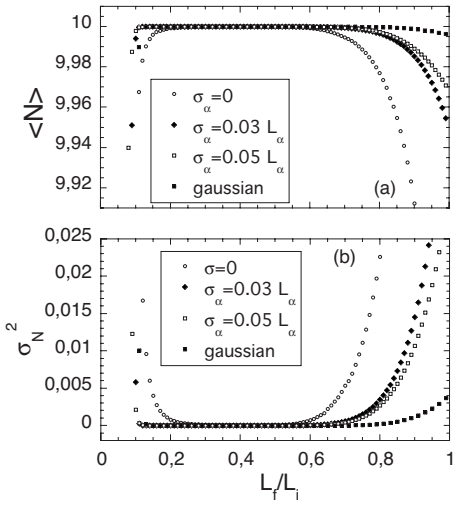


FIG. 4. (a) Asymptotic mean value and (b) variance of the atom-number distribution of a Tonks-Girardeau gas obtained by sudden weakening-squeezing as a function of the width ratio between the final and initial trap $L_f/L_i$. The process is robust for different values of the smoothing parameter $\sigma$. The initial state is assumed to be in the ground state. $U_\alpha$ are chosen such that $C_i=100$ and $C_f=10$; this means that for the bathtub potential $U_i=(100\pi)^2$ and $U_f=(10\pi)^2$, while for the Gaussian, $U_i=(28\pi)^2$ and $U_f=(8\pi)^2$. In all cases we assume $N_i=C_i=100$.

$$L_f \approx \frac{L_i}{2}, \quad V_f \approx \frac{U_f}{U_i}\frac{V_i}{4}, \tag{15}$$

which warrants the preparation of the Fock state $|C_f\rangle$ corresponding to the $U_f$ family.

## V. NONZERO TEMPERATURE

The above formalism can be generalized in a straightforward way to account for the atom-number distribution resulting from an arbitrary initial state at nonzero temperature. It suffices to redefine

$$\hat{\Lambda}_i = \sum_n \pi_n |\varphi_n^i\rangle\langle\varphi_n^i|, \tag{16}$$

which, in general, is not a projector now, where $\pi_n$ is the occupation probability of the state $|\varphi_n^i\rangle$. In the ground state of the TG gas $\pi_n=1 \ \forall n=1,\ldots,N_i$ and $\pi_n=0$ otherwise. For a thermal state, the Fermi-Dirac weights $\pi_n=\{\exp[\beta(E_n^i-\mu)]+1\}^{-1}$ (with $\beta=1/k_BT$ where $k_B$ is the Boltzmann constant and $T$ the absolute temperature) result due to the effective Pauli exclusion principle mimicked by bosons in the TG regime [23]. Notice the normalization $\Sigma_n\pi_n=N_i$. The preparation of a Fock state by a sudden change in the trap will still be feasible as long as $\hat{\Lambda}_f \subseteq \hat{\Lambda}_i$. The numerical results in Fig. 5 (upper panel) illustrate the degradation of the quality of final state with increasing temperature. However, the lower panel shows that this negative effect of temperature can be compensated by starting from a "bigger" initial trap with larger capacity.

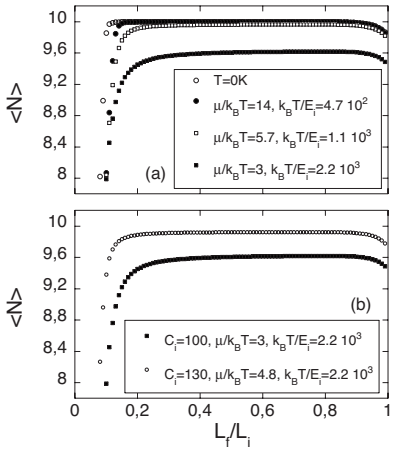


FIG. 5. Effect of temperature on the asymptotic mean value of the atom-number distribution of a Tonks-Girardeau gas obtained by sudden weakening-squeezing for the case of a square potential as a function of the width ratio between the initial and final trap. In plot (a) $U_i=(100\pi)^2$ and $U_f=(10\pi)^2$ so the capacities $C_\alpha$ are the same as in Fig. 4, but the initial occupation is now chosen to be $N_i=0.8C_i$. As we increase the temperature, $\mu/k_BT \leq 5$ the method starts to fail, but this can be improved by increasing the capacity of the initial well, as shown in plot (b), where the weakening-squeezing process is applied at the same temperature but two different initial traps: $U_i=(100\pi)^2$ used in the previous plot and $U_i=(130\pi)^2$, both with $N_i=0.8C_i$ ($E_i=\hbar^2\pi^2/2mL_i^2$).

## VI. DISCUSSION AND CONCLUSION

We conclude that the controlled preparation of atomic Fock states in the strongly interacting (Tonks-Girardeau) regime can be achieved by combining weakening and squeezing of the trapping potential. The process is robust with respect to the smoothness of the potential trap, and moreover the deteriorating effect of increasing temperature can be compensated by enlarging the capacity of the initial trap. However, it is still an experimental challenge to get to the strong TG limit which must translate into a correction to the fidelity. By contrast, noninteracting polarized Fermions would be an ideal system for Fock state preparation. For ultracold fermions, due to the wave-function antisymmetry, *s*-wave scattering is forbidden and generally *p*-wave interactions can be neglected so that the gas is noninteracting to a good approximation. Such type of gases can be prepared in the laboratory with linear densities of the order $0.2-2 \ \mu m^{-1}$ for which the polarization remains constant in a given experiment [26]. For such gases the trap-reduction technique can be directly extended to two and three dimensions.

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